QUESTIONS AND DISCUSSIONS.

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DISCUSSIONS.

In the first discussion this month, Mr. Kazarinoff gives simple demonstrations of several theorems on conic sections in space, suggested by the properties of focal conics of a system of confocal quadrics. Without access to original sources or other adequate references, the author was led to these theorems by direct study of Figs. 1 and 2 of the article. Though Mr. Kazarinoff's theorems lack entire novelty it is hoped that the directness and simplicity of his demonstrations may aid in making them more widely known and admired; and that students of conics may thereby be incited to look into the generalizations of Chasles and Plücker.

The second discussion is of interest as affording an instance of the need of a theorem in mathematics for the purpose of answering a particular question in an applied science. In connection with a condition for stability of thermodynamic equilibrium of a fluid phase of a two-component body Professor Trevor is led to a property of homogeneous functions. The result is itself of interest, and suggests the possibility of generalizations. Probably the denominator $x_3 + x_4$ might be altered to any linear function of the variables with like results. Do similar theorems exist for more general types of transformation? What are the corresponding facts for other isobaric functions than homogeneous? Can the facts in the present and other cases be deduced in any simpler way?

Direct interpretations of imaginary intersections of geometric loci, by operations on the figure itself, have frequently aroused interest. Several such treatments exist for determining complex roots of quadratic and cubic equations. Mr. Mathews gives in the third discussion a unified treatment, by which a number of such questions can be resolved by the application of essentially a single idea—his *virtual image* of a circle with regard to a straight line.

I. DUPIN'S THEOREM.[1]

By D. C. KAZARINOFF, University of Michigan.

It is well known that the consideration of the family of confocal quadrics

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1 \quad (a^2>b^2>c^2), \tag{F}$$

leads to the three focal conics:[2]

[1] Read before the Minnesota Section of the Mathematical Association of America, May 31, 1919.

[2] C. Smith, *An elementary treatise on Solid Geometry*, 10th ed., London, 1905, pp. 144, 145.

Ellipse:
$$\frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} = 1, \qquad z = 0, \tag{E}$$

Hyperbola:
$$\frac{x^2}{a^2 - b^2} + \frac{z^2}{c^2 - b^2} = 1, \qquad y = 0, \tag{H}$$

Imaginary conic:
$$\frac{y^2}{b^2 - a^2} + \frac{z^2}{c^2 - a^2} = 1, \qquad x = 0. \tag{I}$$

Furthermore the following property is well known:[1]

The locus of the vertices of the right circular cones which envelop the surfaces (F) *consists of the focal conics* (E), (H), *and* (I).[2]

Since the part of the plane $z = 0$ interior to (E) and the part of the plane $y = 0$ which does not contain the center and is bounded by (H) may both be regarded as limiting cases of surfaces (F),[3] the following particular property may be derived:

Property (A) (see Fig. 1): *Given any ellipse* (E) *and any hyperbola* (H) *in perpendicular planes, so*

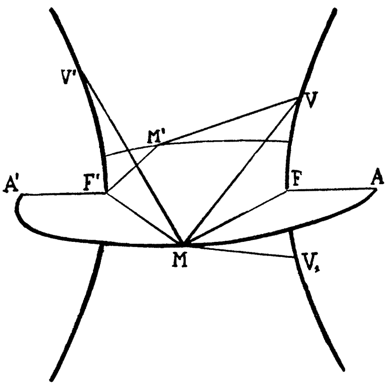

FIG. 1.

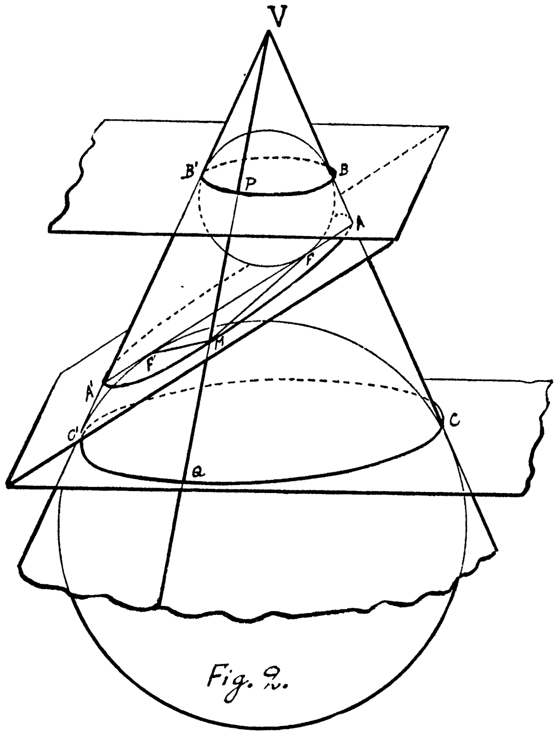

FIG. 2.

placed that the foci of (H) *are the vertices of* (E), *and vice versa. Then the locus of the vertices of right circular cones on the elliptic base* (E) *is the hyperbola* (H), *and vice versa.*[4]

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[1] C. Smith, *idem*, pp. 153–155. See also G. Salmon, *A treatise on conic sections*, 3d ed., London, 1855, p. 303.

[2] This result is due to Steiner, *Journal für die reine und angewandte Mathematik*, vol. 1 (1826), pp. 47 ff.—EDITOR-IN-CHIEF.

[3] C. Smith, *idem*, pp. 144, 145.

[4] This result is due to C. Dupin, *Correspondance sur l'école polytechnique*, vol. 2 (Jan., 1813), p. 424; in the *Correspondance* for April, 1804, Hachette attributed the result to Dupin.—EDITOR-IN-CHIEF.

My purpose is to present here an elementary proof of the property (A) and to deduce from it some other reciprocal properties of (E) and H.

Proof of property (A) (see Fig. 2):

Let V be the vertex of a right circular cone on the elliptic base $AA'M$ of which the vertices and the foci are respectively A, A' and F, F'. The points of contact of two spheres inscribed in the cone and touching the plane $AA'M$ will be the foci F, F'.[1]

The plane VAA' passes through the centers of the spheres and is therefore perpendicular to the plane $AA'M$.

Furthermore,

$$VA' - VA = B'A' - BA = A'F - AF = FF'.$$

Hence V is a point on an hyperbola with vertices and foci respectively at F, F' and A, A' in a plane perpendicular to the plane $AA'M$.[2] Since it can be proved in a similar way that from every point of the above hyperbola the ellipse $AA'M$ will be projected by a right circular cone, we have (Fig. 1):

The locus of the point V is an hyperbola with the same relative position to the ellipse $AA'M$ as (H) has with respect to (E).

Mutatis mutandis we can prove the second part of the property (A).

From Fig. 2 we have also:

$$MV + MF' = MV + MQ = VQ = \text{constant}, \tag{b}$$

$$MV - MF = MV - MP = VP = \text{constant}. \tag{c}$$

Property (B) (See Fig. 1):

V, V' being any two fixed points on different branches of the hyperbola (H), and M any point on the ellipse (E),

$$MV + MV' = \text{constant}.$$

Proof:

$$MV + MF' = \text{constant},$$

$$MV' + MF = \text{constant},$$

$$MF' + MF = AA' = \text{constant}.$$

Hence

$$MV + MV' = \text{constant}.$$

Property (C):

V, V_1 being any two fixed points on the same branch of (H), and M any point on the ellipse (E),

$$MV - MV_1 = \text{constant}.$$

Proof:

[1] G. Salmon, *idem*, footnote on p. 305.

[2] This property is due to G. P. Dandelin, *Nouv. Mém. Acad. Bruxelles*, volume 2 (1822), p. 172. —Editor-in-Chief.

$$MV - MF = \text{constant.}$$

$$MV_1 - MF = \text{constant.}$$

Hence

$$MV - MV_1 = \text{constant.} \qquad \text{Q. E. D.}$$

Property (D):

M, M' being any two fixed points on the ellipse (E), and V any point on the hyperbola (H),[1]

$$VM - VM' = \text{constant.}$$

Proof:

$$MV + MF' = M'V + M'F' = \text{constant.}$$

Hence

$$VM - VM' = \text{constant.} \qquad \text{Q. E. D.}$$

It can be proved without difficulty that the locus of the points having the property (B) is the hyperbola (H), as well as that the locus of the points having the property (C) is the ellipse (E).

II. A Property of Homogeneous Functions.

By J. E. Trevor, Cornell University.

When a body constituted of two independent component substances and subject to no mechanical or thermal separation of its parts is in a state of thermodynamic equilibrium, the body may exhibit distinct liquid or aëriform parts. Any such part has variable mass, composition, and thermodynamic state, and is termed a fluid "phase" of the body. When x_1, x_2, x_3, x_4 denote the volume, the entropy, and the component-masses of a two-component phase, the energy of the phase is a homogeneous function $E(x_1, x_2, x_3, x_4)$ such that

$$t\cdot E(x_1, x_2, x_3, x_4) = E(tx_1, tx_2, tx_3, tx_4), \tag{1}$$

where t is any positive number.[2] The specific volume, specific entropy, and specific component-masses of the body are defined by the equations

$$y_1 = \frac{x_1}{x_3 + x_4}, \quad y_2 = \frac{x_2}{x_3 + x_4}, \quad y_3 = \frac{x_3}{x_3 + x_4}, \quad y_4 = \frac{x_4}{x_3 + x_4} = 1 - y_3.$$

When $t = 1/(x_3 + x_4)$, and X is written for $x_3 + x_4$, the equation (1) becomes

$$E = X\cdot E(y_1, y_2, y_3, 1 - y_3)$$

$$= X\cdot e(y_1, y_2, y_3), \tag{2}$$

where the "specific energy" e of the phase is a function of the three variables y_1, y_2, y_3.

[1] Properties (B), (C) and (D) are due to Dupin, *Correspondance sur l'école polytechnique*, Jan., 1807, p. 218, and Jan., 1813, p. 424. For further references in this connection, and to generalizations by Chasles and Plücker, the *Encyclopédie des Sciences Mathématiques*, tome III, vol. 4, pp. 81–86, and tome III, vol. 3, pp. 51–52, may be consulted.—Editor-in-Chief.

[2] Functions which possess this restricted form of homogeneity are termed "positively homoneous" by Bolza, *Lectures on the Calculus of Variations*, Chicago, 1904, p. 119.—Editor.